Journal of Statistical Physics, Vol. 35, Nos. 1/2, 1984

Mean Field Bound and GHS Inequality

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Received September 19, 1983

A new proof of the mean field bounds for magnetizations is presented. It applies to any single-component spin system which allows GHS inequality, and to an N-vector model for $N \geqslant 3$, and to an N-solid sphere model for all values of N, provided that the interactions are ferromagnetic and translation invariant.

KEY WORDS: GHS inequality; mean field bound; unbounded spin systems; N-vector model.

1. INTRODUCTION

The rigorous statistical mechanics of classical spin systems is one of the few fields of physics in which one can deal with the infinite degrees of freedom in a mathematically sensible manner. In fact, the recent developments in the field (especially in the theory of phase transition) have brought us many highly sophisticated nonperturbative analyses, both qualitative and quantitative.[(1–3)] In spite of all these successes, we cannot underrate the worth of the most crude one-body approximation, i.e., mean field theory, since it still provides us a good qualitative picture of the phase transition.

Here, we concentrate on the rigorous relations between the mean field approximation and corresponding true behavior of the system. One of them is a series of inequalities stating that the mean field critical temperature gives an upper bound of the real critical temperature. Another one, a subject of this paper, is the mean field bound for the magnetization. It

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0022-4715/84/0400-0099$03.50/0

states that the mean field magnetization is always not less than the real magnetization. Such an inequality was first proved for spin-1/2 Ising ferromagnets by Thompson.[4] And recently, Pearce[5] extended this result to some class of single-component bounded spin systems, and to the N-vector models with $N = 2, 3$. See also Ref. 6 and the references therein for the related subjects.

In this paper, we give a new proof of the mean field bounds for magnetizations, based on the GHS inequality. Our result is an extension of Thompson's, and it is complementary to that of Pearce's. In particular, our method applies to all the single-component spin systems (where spins can be bounded or unbounded) which allow the GHS inequality (Section 3). As for the multicomponent spin systems, our result is less complete. We prove the bounds for the N-vector models with $N \geqslant 3$, and for the N-solid sphere model with an arbitrary value of N (Section 4).

Section 2 deals with formal definitions. Our new method appears in Section 3. We believe our method is the simplest and the most intuitive among the three.

After this work was submitted, we learned from Professors A. D. Sokal and M. Aizenman that our main result was already found by Professor C. M. Newman, and was reported on several times.[17-19] His elegant method is based on inequality (10), the equality $\partial m_{\mathrm{MF}}/\partial\beta - Jm_{\mathrm{MF}}\partial m_{\mathrm{MF}}/\partial H = 0$, and the initial condition $m(0, H) = m_{\mathrm{MF}}(0, H)$. By the simple technique of the characteristic curve, these lead to the inequality (8). We are grateful to Professors Charles M. Newman, Alan D. Sokal, and Michael Aizenman for their kind correspondences.

2. MODELS AND MEAN FIELD THEORIES

We first describe our model systems. Let Λ be an arbitrary lattice invariant under translations,[3] for example $\Lambda = Z^d$, or T^d (d-dimensional torus). Spins $\boldsymbol{\phi}_x \in R^N$ are associated to each $x \in \Lambda$, with *a priori* measure $d\nu(\boldsymbol{\phi})$.

Properties of the system are characterized by a Hamiltonian of the form

$$\mathscr{H} = -\frac{\beta}{2}\sum_{x,y} J_{xy}\boldsymbol{\phi}_x \cdot \boldsymbol{\phi}_y - H\sum_x \phi_x^{(1)} \tag{1}$$

[3] Actually, the "translation" can be any bijection on Λ. This allows a wide class of lattices (Bethe lattice, for example).

where $\phi_x^{(i)}$ denotes the ith component of $\boldsymbol{\phi}_x$,

$$\boldsymbol{\phi}_x \cdot \boldsymbol{\phi}_y = \sum_{i=1}^{N} \phi_x^{(i)} \phi_y^{(i)}$$

and we assume that $H \geqslant 0$, $J_{xy} = J_{yx} \geqslant 0$, J_{xy} is invariant under any lattice translation, and $J = \sum_y J_{xy} < \infty$.

The Gibbs state for Hamiltonian (1) is formally given by

$$\langle \cdots \rangle = \int \prod_{x \in \Lambda} d\nu(\boldsymbol{\phi}_x) e^{-\mathscr{H}} (\cdots) \Big/ \int \prod_{x \in \Lambda} d\nu(\boldsymbol{\phi}_x) e^{-\mathscr{H}} \tag{2}$$

For infinite Λ, $\langle \cdots \rangle$ is defined as a limit of suitable sequence of the states for finite sublattices.[7,8] (Such a state is called a "limit Gibbs state.") For the reason which will become clear in Section 3, we consider only the translation-invariant states in the following. (It is always possible to take such states.)

The special expectation value we consider is the magnetization (or order parameter) for given β, H, which is defined by

$$m(\beta, H) = \langle \phi_x^{(1)} \rangle \tag{3}$$

And the spontaneous magnetization at β is

$$m_s(\beta) = \lim_{H \to +0} m(\beta, H) \tag{4}$$

For a wide class of these models [for example, Ising and ϕ^4 models for $d \geqslant 2$, isotropic N-component models ($N \geqslant 2$) for $d \geqslant 3$, if $\Lambda = Z^d$], it has been rigorously shown that they undergo phase transitions.[2,7,9] That is, there exists a positive β_c (inverse critical temperature) such that $m_s(\beta) = 0$ for $\beta < \beta_c$ and $m_s(\beta) > 0$ for $\beta > \beta_c$.

The mean field theory provides a simple picture of these phenomena. Mean field magnetization $m_{\mathrm{MF}}(\beta, H)$ is defined as a maximum solution of the following self-consistent equation:

$$m_{\mathrm{MF}} = \frac{\langle \phi^{(1)} \exp\{(H + \beta J m_{\mathrm{MF}}) \phi^{(1)}\} \rangle_0}{\langle \exp\{(H + \beta J m_{\mathrm{MF}}) \phi^{(1)}\} \rangle_0} \tag{5}$$

where $J = \sum_y J_{xy}$ and

$$\langle \cdots \rangle_0 = \int (\cdots) \, d\nu(\boldsymbol{\phi}) \Big/ \int d\nu(\boldsymbol{\phi}) \tag{6}$$

This $m_{\mathrm{MF}}(\beta, H)$ also shows the phase-transition-like behavior. Its "inverse critical temperature" β_{MF} is given by

$$\beta_{\mathrm{MF}} J \langle (\phi^{(1)})^2 \rangle_0 = 1 \tag{7}$$

3. SINGLE-COMPONENT SYSTEMS

Consider the single-component systems (i.e., $N = 1$) with *a priori* measure $d\nu(\phi)$ of the following types.[4]

(i) Spin-$n/2$ Ising model: $d\nu(\phi) = [1/(n+1)]\sum_{j=0}^{n}\delta(-n+2j+\phi)\, d\phi$.

(ii) Continuous bounded spin systems: $d\nu(\phi) = g(\phi)\, d\phi$, where $\operatorname{supp} g = [-a, a]$ $(0 < a < \infty)$, $g(\phi)$ is continuously differentiable and strictly positive for $\phi \in (-a, a)$, and $(g'(\phi)/g(\phi))$ is concave on $[0, a)$.

(iii) ϕ^4-like (unbounded spin) systems: $d\nu(\phi) = e^{-V(\phi)}\, d\phi$ where $V(\phi) = \sum_{i=1}^{M} a_i \phi^{2i}$, $M \geqslant 2$, a_1 real, $a_i \geqslant 0$ for $i \geqslant 2$, and $a_M \neq 0$ (and any well-defined limit of these measures).

Then, the main result of this section is as follows.

Theorem 1. For a single-component system with *a priori* measure of the type (i)–(iii), we have the following mean field bound for magnetization:

$$m(\beta, H) \leqslant m_{\mathrm{MF}}(\beta, H) \tag{8}$$

for arbitrary positive (or vanishing) β, H.

We are going to prove this theorem in the remainder of this section. First, note that the GHS inequality

$$\begin{aligned} U_3(x, y, z) = \langle \phi_x \phi_y \phi_z \rangle - \langle \phi_x \rangle \langle \phi_y \phi_z \rangle - \langle \phi_y \rangle \langle \phi_z \phi_x \rangle - \langle \phi_z \rangle \langle \phi_x \phi_y \rangle \\ + 2\langle \phi_x \rangle \langle \phi_y \rangle \langle \phi_z \rangle \leqslant 0 \qquad \text{for} \quad x, y, z \in \Lambda \end{aligned} \tag{9}$$

is known to be valid for all the systems considered in the Theorem 1.[10–12] This inequality, with the translation invariance, immediately implies the following relation[13]:

$$\frac{\partial m(\beta, H)}{\partial \beta} - J m(\beta, H) \frac{\partial m(\beta, H)}{\partial H} \leqslant 0 \tag{10}$$

The basic idea of our proof is to integrate the "differential inequality" (10) to get the desired inequality (8). In order to perform the "integration," we introduce our "generalized self-consistent equation."

Definition 2. For any $\beta, H \geqslant 0$ and $x \in [0, 1]$, $m^*(x; \beta, H)$ is a maximum solution of the equation

$$m^* = m(\beta(1-x), H + \beta J x m^*) \tag{11}$$

[4] In this section, we use ϕ_x for spin variables (instead of $\boldsymbol{\phi}_x$). Hamiltonian (1) now becomes

$$\mathscr{H} = -(\beta/2)\sum_{x,y} J_{xy}\phi_x\phi_y - H\sum_x \phi_x$$

where $m(\beta,H)$ [defined in Eq. (3)] is regarded as a given function of β and H.

This definition may seem to be too abstract, since we do not know the function $m(\beta,H)$ explicitly. But we actually have the following.

Lemma 3. For $\beta,H \geqslant 0$ and $x \in [0,1]$, $m^*(x;\ \beta,H)$ exists.

Proof. Fix β,H,x and rewrite Eq. (11) as $m^* = F(m^*)$. The existence of the maximum solution follows from the following properties of $F(m^*)$. For $m^* \geqslant 0$, (a) $F(m^*) \geqslant 0$, (b), $F'(m^*) \geqslant 0$, (c) $F''(m^*) \leqslant 0$, and (d) $F'(m^*) \to 0$ as $m^* \to \infty$. The properties (a), (b), and (c) are the consequences of Griffiths I, II[14] and GHS inequalities. The property (d) is trivial for the bounded spin systems [measures of type (i), (ii)]. For the unbounded spin systems [measure (iii)], (d) follows from the following upper bound for the free energy per site:

$$\int_0^H m(\beta,H')\,dH' = -f(\beta,H) = \frac{1}{|\Lambda|}\ln\int\prod_x d\nu(\phi_x)\,e^{-\mathscr{H}}$$
$$\leqslant \ln\int d\nu(\phi)\exp\left[\beta J\phi^2/2 - V(\phi) + H\phi\right]$$
$$\sim H^{[1/(2M-1)]+1} \qquad \text{as} \quad H \to \infty$$

where we used a trivial upper bound $\phi_x\phi_y \leqslant (\phi_x^2 + \phi_y^2)/2$. ■

Now we prove the increasing property of $m^*(x;\ \beta,H)$.

Lemma 4. For any $\beta,H \geqslant 0$, we have

$$\frac{\partial}{\partial x} m^*(x;\ \beta,H) \geqslant 0 \tag{12}$$

Proof. We only sketch the outline of the proof, since it consists of rather tedious estimates. If we differentiate (11) by x, we have

$$\left[1 - \beta Jx\frac{\partial}{\partial\tilde{H}}m(\tilde{\beta},\tilde{H})\right]\frac{\partial}{\partial x}m^*(x;\ \beta,H)$$
$$= -\beta\left[\frac{\partial}{\partial\tilde{\beta}}m(\tilde{\beta},\tilde{H}) - Jm(\tilde{\beta},\tilde{H})\frac{\partial}{\partial\tilde{H}}m(\tilde{\beta},\tilde{H})\right]$$

where $\tilde{\beta} = \beta(1-x)$, $\tilde{H} = H + \beta Jxm^*(x;\ \beta,H)$. The right-hand side is nonnegative by (10), and also is the prefactor of the left-hand side. This follows from the inequality $\partial m^*(x;\ \beta,H)/\partial H \geqslant 0$, which is a consequence of Eq. (10) and Griffiths II inequality. ■

Proof of Theorem 1. It follows from the definition (11) that $m^*(0; \beta, H) = m(\beta, H)$ and $m^*(1; \beta, H) = m_{\mathrm{MF}}(\beta, H)$. Thus mean field bound (8) follows by integrating[5] Eq. (12). ■

Observe that, when we increase the parameter x from 0 to 1, the effective inverse temperature $\tilde{\beta} = \beta(1-x)$ in the "generalized self-consistent equation" (11) gradually decreases until it finally becomes 0 (i.e., infinite temperature). While, the effective field $\tilde{H} = H + \beta J x m^*$ increases to compensate for this change of temperature. This mechanism of "temperature shifting" enables us to compare a system of infinite degrees of freedom with an essentially one-bodied system.

Finally, we note that the mean field bound (8) implies the simple lower bound for inverse critical temperature:

$$\beta_c \geqslant \beta_{\mathrm{MF}}$$

4. MULTICOMPONENT SYSTEMS

In this section, we follow the idea of Pearce to establish the mean field bounds for multicomponent systems. The central idea is to compare them with corresponding single-component systems.

Suppose we are dealing with the N-component system defined by Hamiltonian (1). We now define the single-component system corresponding to (1) by

$$\mathscr{H}_1 = -\frac{\beta}{2}\sum_{x,y} J_{xy}\sigma_x\sigma_y - H\sum_x \sigma_x \tag{13}$$

where σ_x's are single-component spins with *a priori* measure,

$$d\mu(\sigma) = \int d\nu(\phi)\,\delta(\phi^{(1)} - \sigma) \tag{14}$$

We denote the Gibbs state corresponding to (13), (14) as $\langle \cdots \rangle_1$. Note that this single-component system is obtained by eliminating all the interactions in Hamiltonian (1) except the first component part, and then integrating out the other components of spins. Since the magnetic field H is along the first component, the corresponding mean field magnetizations for these two systems have exactly the same values. For the "true" magnetizations, we restate the following lemma due to Pearce.[5,15]

[5] For the systems without Lee–Yang properties,[3,7] one might worry about the differentiability of $m^*(x)$. In such cases, we first take Λ as a finite lattice, and take the limit $|\Lambda| \to \infty$ after establishing the inequality (8).

Lemma 5. For the N-component model (1), with *a priori* measure

$$d\nu(\phi) = e^{-U(\phi\cdot\phi)} d^N\phi, \qquad U(x) = \sum_{i=1}^{M} a_i x^i \qquad a_1 \text{ real}, \quad a_i \geqslant 0 \quad \text{for} \quad i \geqslant 2 \tag{15}$$

and the corresponding single-component model (13), (14), we can state

$$\langle \sigma_x \rangle_1 \geqslant \langle \phi_x^{(1)} \rangle \tag{16}$$

Proof. Let

$$\tilde{\mathscr{H}} = -\beta/2 \sum_{x,y} J_{x,y} \phi_x^{(1)} \phi_y^{(1)}$$

$$-\beta'/2 \sum_{x,y} J_{x,y} \sum_{i=2}^{N} \phi_x^{(i)} \phi_y^{(i)} - H \sum_x \phi_x^{(1)}$$

[Note that (1) and (13) correspond to $\beta' = \beta$ and $\beta' = 0$.] We consider duplicated sets of spins $\{\phi_x\}$ and $\{\phi'_x\}$. Let $\langle \cdots \rangle_{\text{dup}}$ be the Gibbs state obtained from the Hamiltonian $\tilde{\mathscr{H}}(\{\phi_x\}) + \tilde{\mathscr{H}}(\{\phi'_x\})$. As usual,[3,12] we define,

$$\alpha_x = \left(\phi_x^{(1)} + \phi_x'^{(1)}\right)/\sqrt{2}, \qquad \beta_x = \left(\phi_x^{(1)} - \phi_x'^{(1)}\right)/\sqrt{2}$$

$$\gamma_x^{(i)} = \left(\phi_x^{(i)} + \phi_x'^{(i)}\right)/\sqrt{2}, \qquad \delta_x^{(i)} = \left(\phi_x'^{(i)} - \phi_x^{(i)}\right)/\sqrt{2}$$

for $i = 2, 3, \ldots, N$.

We want to show that

$$\langle \alpha^A \beta^B \sum_{i=2}^{N} \gamma^C \sum_{i=2}^{N} \delta^D \sum_{i=2}^{N} \gamma^E \delta^F \rangle_{\text{dup}} \geqslant 0$$

where $A, B, \ldots$ are arbitrary index sets. For this, we note the following (i) The Hamiltonian $\tilde{\mathscr{H}}(\{\phi_x\}) + \tilde{\mathscr{H}}(\{\phi'_x\})$ is still a ferromagnetic 2-form in terms of α, β, ... variables. (ii) The measure is

$$U(\phi\cdot\phi) + U(\phi'\cdot\phi') = \text{even} - \sum_{k,l} C(k,l)(\alpha\beta)^k \Big(\sum_i \gamma^{(i)}\delta^{(i)}\Big)^l$$

where $C(k,l) \geqslant 0$, and terms denoted "even" are invariant under the changes of the signs of α, β and $\gamma^{(i)}, \delta^{(i)}$ (simultaneously in $i = 2, 3, \ldots, N$). Expanding the exponentials in the integral, and using (i), (ii), we are led to the claimed nonnegativity. So we can state that

$$\frac{\partial}{\partial \beta'} \langle \phi_x^{(1)} \rangle = \sum_{y,z} J_{yz} \left\langle \phi_x^{(1)} \sum_{i=2}^{N} \left(\phi_y^{(i)} \phi_z^{(i)} - \phi_y'^{(i)} \phi_z'^{(i)} \right) \right\rangle_{\mathrm{dup}}$$

$$= -\frac{1}{\sqrt{2}} \sum_{y,z} J_{yz} \left\langle (\alpha_x + \beta_x) \sum_{i=2}^{N} \left(\gamma_y^{(i)} \delta_z^{(i)} + \gamma_z^{(i)} \delta_y^{(i)} \right) \right\rangle_{\mathrm{dup}} \leqslant 0$$

which proves the lemma. ■

So, if we could bound $\langle \sigma_x \rangle_1$ by its mean field value, we can state the mean field bound of the original N-component system (1).

In the following, we restrict ourselves to two types of measures, which can be obtained as limiting cases of (15).

(A) *N-vector model* [or $O(N)$-Heisenberg model]:

$$d\nu(\phi) = \delta(\phi \cdot \phi - 1)\, d^N \phi \tag{17}$$

This measure is derived by

$$\lim_{\lambda \to \infty} e^{-U(\phi \cdot \phi)}\, d^N \phi, \qquad U(x) = \lambda (x-1)^2.$$

(B) *N-solid sphere model*:

$$d\nu(\phi) = \begin{cases} d^N \phi, & \text{for} \quad \phi \cdot \phi \leqslant 1 \\ 0, & \text{for} \quad \phi \cdot \phi > 1 \end{cases} \tag{18}$$

This is, $\lim_{n \to \infty} e^{-U(\phi \cdot \phi)}\, d^N \phi$, $U(x) = x^{2n}$.

Now, we consider the corresponding single-component systems for (A) and (B). In these cases, *a priori* measures $d\mu(\sigma)$'s for single-component spin can be explicitly calculated:

$$\text{(A)} \qquad d\mu(\sigma) = (1 - \sigma^2)^{(N-3)/2}\, d\sigma, \qquad \sigma \in [-1, 1] \tag{19}$$

$$\text{(B)} \qquad d\mu(\sigma) = (1 - \sigma^2)^{(N-1)/2}\, d\sigma, \qquad \sigma \in [-1, 1] \tag{20}$$

It is easy to check that these $d\mu(\sigma)$'s are measures of type (ii) (see Section 3), for $N \geqslant 3$ [case (A)], and for all $N \geqslant 1$ [case (B)]. Thus we are automatically led to the following:

Proposition 6. For the N-vector model (A) with $N \geqslant 3$, and the N-solid sphere model (B) with $N \geqslant 1$, we have the mean field bounds i.e.

$$m(\beta, H) \leqslant m_{\mathrm{MF}}(\beta, H) \quad \text{and} \quad \beta_c \geqslant \beta_{\mathrm{MF}}$$

Remarks. (1) It is easy to see that for the N-vector model (A),

$$\beta_{\mathrm{MF}} = 1/JN$$

This implies the well-known[16] lower bound for β_c:

$$\beta_c \geqslant 1/JN$$

(2) Pearce[5] proved the mean field bound of the N-vector model for $N \leqslant 3$. So the bound for this model has been proved for all values of N. It is quite interesting that Pearce's method and ours work in complementary regions of N. (After this work was submitted, Ref. 20 appeared in this journal. It contains more results about the N-vector model.)

ACKNOWLEDGMENTS

We would like to thank Professor Masuo Suzuki for useful discussions and a critical reading of the manuscript. H. T. expresses his gratitude to Professor Ryogo Kubo for stimulating conversations. T. H. is grateful to Professors Masaaki Kato and Tamiaki Yoneya for continuous encouragements.

REFERENCES

1. M. Aizenman, *Commun. Math. Phys.* **86**:1 (1982).
2. J. Fröhlich, R. Israel, E. H. Lieb, and B. Simon, *Commun. Math. Phys.* **62**:1 (1978).
3. J. Glimm and A. Jaffe, *Quantum Physics—A Functional Integral Point of View* (Springer, New York, 1981).
4. C. J. Thompson, *Commun. Math. Phys.* **24**:61 (1971).
5. P. A. Pearce, *J. Stat. Phys.* **25**:309 (1981).
6. A. D. Sokal, *J. Stat. Phys.* **28**:431 (1982).
7. D. Ruelle, *Statistical Mechanics—Rigorous Results* (Benjamin, New York, 1969).
8. R. B. Griffiths, in *Statistical Mechanics and Quantum Field Theory*: 1970 Les Houches, C. DeWitt and R. Stora, eds. (Gordon & Breach, New York, 1971).
9. J. Fröhlich, B. Simon, and T. Spencer, *Commun. Math. Phys.* **50**:79 (1976).
10. R. B. Griffiths, C. A. Hurst, and S. Sherman, *J. Math. Phys.* **11**:790 (1970).
11. R. S. Ellis, J. L. Monroe, and C. M. Newman, *Commun. Math. Phys.* **46**:167 (1976).
12. G. S. Sylvester, *J. Stat. Phys.* **15**:327 (1976).
13. A. D. Sokal, *J. Stat. Phys.* **25**:25 (1981).
14. R. B. Griffiths, *J. Math. Phys.* **8**:478, 484 (1967).
15. J. L. Monroe and P. A. Pearce, *J. Stat. Phys.* **21**:615 (1979).
16. B. Simon, *J. Stat. Phys.* **22**:491 (1980).
17. C. M. Newman, Shock Waves and Mean Field Bounds (announcement of results, University of Arizona, 1981).
18. C. M. Newman, Talk at 46th Statistical Mechanics Meeting (Rutgers, New Jersey, 1981); see *J. Stat. Phys.* **27**:836 (1982).
19. C. M. Newman, Talk at Functional Integral Method Workshop (Santa Barbara, California, 1982).
20. J. Slawny, *J. Stat. Phys.* **32**:375 (1983).